

SCCUR – Southern California Conference for Undergraduate Research
13th Annual SCCUR Conference at Riverside, California - November 19, 2005

Markov Analysis and its Application to Reliability Evaluation

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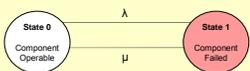
Abstract

- Many engineering systems are subject to failure after a given amount of time, and just how long a system will not fail depends on its design. Thus, a client of the Statistical Consulting Collaboratory requested an analysis of the reliability of his telecommunications system.
- One very useful technique in finding the reliability of a system is Markov modeling.
- This technique involves analytically finding the probability of the system being in each of its potential states, and when summed appropriately these state probabilities lead to the overall reliability of a system.
- However, for systems with a large number of states, solving for the reliability becomes increasingly tedious and complex.
- This facilitates the need for software tools such as Relx Software to handle the systems' reliability.

Background

Continuous-Time Markov Processes:

- Markov Analysis entails using a state diagram to model the various states a system can undergo, or transition to; these models are then used to find the probability of being in each state based on transition rates between states.
- These various states can be 'good' or 'up' states as well as 'bad' or 'down' states.
- The *Reliability function, R(t)*, of a system is the probability as a function of time, *t*, that the system remains operable for at least *t* units of time.
- The *Availability function, A(t)*, of a system is the probability as a function of time, *t*, that the system is operable at time *t*.



- An example of a 2-state model (λ and μ are failure and repair rates, respectively)

Illustration with a 3-State Model:

- Consider a system in which there are two components that are identical, such as a sound system with two speakers.



- We want the system's Availability in this case.
- We need to solve for the state probabilities, which can be found using a differential equations method. The setup is as follows:

$$P_0'(t + dt) = P_0(t)(1 - 2\lambda dt) + P_1(t)\mu dt$$

$$P_1'(t + dt) = P_1(t)2\lambda dt + P_1(t)(1 - \lambda dt - \mu dt) + P_2(t)2\mu dt$$

$$P_2'(t + dt) = P_2(t)\lambda dt + P_2(t)(1 - 2\mu dt)$$

dt is a small enough change in time such that the probability of two or More events occurring during this time interval is negligible.

Solving for Time Dependent solutions
Illustration with a 3-State Model:

- From the equations in the setup, we then let *dt* go to 0, which gives:

$$P_0'(t) = -P_0(t)2\lambda + P_1(t)\mu$$

$$P_1'(t) = P_1(t)2\lambda - P_1(t)(\lambda + \mu) + P_2(t)$$

$$P_2'(t) = P_2(t)\lambda - P_2(t)2\mu$$

- Using Laplace Transforms yields a set of simultaneous linear equations:

$$(s + 2\lambda)P_0(s) = 1 + \mu P_1(s)$$

$$(s + \lambda + \mu)P_1(s) = 2\lambda P_0(s) + 2\mu P_2(s)$$

$$(s + 2\mu)P_2(s) = \lambda P_1(s)$$

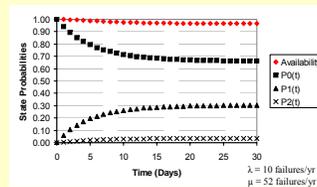
- These can be solved in the *s*-domain and transformed into the *t*-domain to give the state probabilities:

$$P_0(t) = \frac{\mu^2}{(\lambda + \mu)^2} + \frac{2\lambda\mu}{(\lambda + \mu)^2}e^{-(\lambda + \mu)t} + \frac{\lambda^2}{(\lambda + \mu)^2}e^{-2(\lambda + \mu)t}$$

$$P_1(t) = \frac{2\lambda\mu}{(\lambda + \mu)^2} + \frac{2\lambda(\lambda - \mu)}{(\lambda + \mu)^2}e^{-(\lambda + \mu)t} - \frac{2\lambda^2}{(\lambda + \mu)^2}e^{-2(\lambda + \mu)t}$$

$$P_2(t) = \frac{\lambda^2}{(\lambda + \mu)^2} - \frac{2\lambda^2}{(\lambda + \mu)^2}e^{-(\lambda + \mu)t} + \frac{\lambda^2}{(\lambda + \mu)^2}e^{-2(\lambda + \mu)t}$$

Graph of State Probabilities vs. Time:



- The Availability is the sum of the good states. Note how swiftly the state probabilities converge.
- Setting *t* to infinity, we can find the system's limiting state probabilities.

$$P_0(\infty) = \frac{\mu^2}{(\lambda + \mu)^2}$$

$$P_1(\infty) = \frac{2\lambda\mu}{(\lambda + \mu)^2}$$

$$P_2(\infty) = \frac{\lambda^2}{(\lambda + \mu)^2}$$

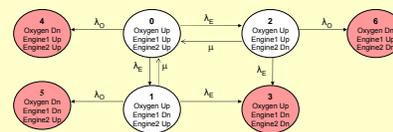
Objectives

- The purpose of the project was to create a slide tutorial with PowerPoint to illustrate the concepts needed to understand Markov analysis.
- The analytic portion explicates many realistic systems, from state diagrams to time dependent solutions; this information is the foundation of Relx
- For larger Markov models it is necessary to rely on Relx Software, which efficiently outputs similar data as calculated analytically [by hand] using Markov analysis.

Methods

Solving for Time Dependent solutions
Illustration with a 7-State Model:

- A slightly more realistic model in terms of size and complexity would be a shuttle model with three components: oxygen and two identical engines.



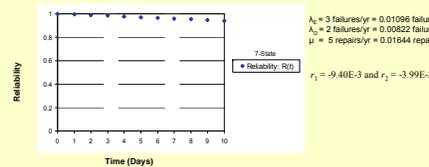
- This system's Reliability function (the sum of the white or good states) is as follows (which is solved in a similar way to the 3-State model):

$$R(t) = P_0(t) + P_1(t) + P_2(t)$$

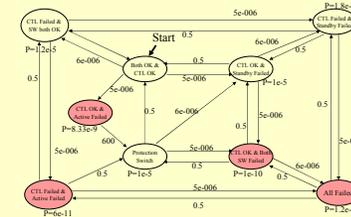
$$R(t) = \frac{r_1 + \lambda_O + 3\lambda_E + \mu}{(r_1 - r_2)} e^{r_1 t} + \frac{r_2 + \lambda_O + 3\lambda_E + \mu}{(r_2 - r_1)} e^{r_2 t}$$

$$r_1, r_2 = \frac{-(3\lambda_E + 2\lambda_O + \mu) \pm \sqrt{\lambda_E^2 + 6\lambda_E\mu}}{2}$$

- In this case we are interested in the Reliability instead of Availability because the red states are catastrophic failures from which there would be no repairs.



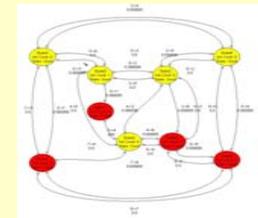
Solving for the Telecommunications System's Reliability with Relx



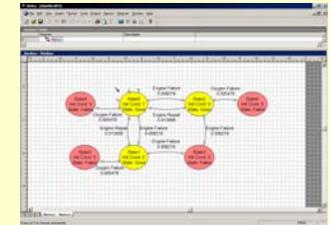
- Relx features a workspace in which the user can insert oval states, which are then initialized to good or bad states.
- The user can then click on and drag arrows from state to state. These are transitions, and the user can enter transition rate values for the system.
- After the model is created, the user can go to Project => Calculate, to yield results for reliability, availability, etc.

Results

Telecommunications Model in Relx



Shuttle Model in Relx



Sequential Data Generated Outputs

7-State Shuttle Model

Telecomm. Model

- Note that the outputted reliability values for the shuttle are exactly the same as the values calculated and plotted in the graph from before.

Conclusion

- There are many ways to calculate system reliability, and one very effective technique is through Markov modeling.
- This method incorporates differential equations into the solving of Reliability, Availability, steady state and time dependent probabilities to the various states in a Markov model.
- Markov analyses' challenges are that the differential equations become very difficult to solve by hand when the systems become large and complex, such as real systems today. Therefore, it is necessary to use powerful software tools, such as Relx software, to solve larger Markov models.